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Affine Toda field theory from tree unitarity

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Abstract. The elasticity property (i.e. no particle creation) is used in the tree level scattering of scalar particles in $1+1$ dimensions to construct affine Toda field theory (ATFT) associated with the root systems of the groups $a_2^{(2)}$ and $c_2^{(1)}$. A general prescription is given for constructing ATFT (associated with rank two root systems) with two self-conjugate scalar fields. It is conjectured that the same method could be used to obtain the other ATFT associated with higher rank root systems.

1 Introduction

The present note is motivated by the opening section of the paper by Dorey in [1], which in turn was inspired by a remark in an article by Goebel on the sine-Gordon S-matrix [2].

The aim of this paper is the construction of affine Toda field theory (ATFT) from well-known scalar field theory by demanding the elasticity property (i.e. no particle production) in the scattering of particles at tree level. We show that the tree level calculation would suffice for this purpose. Once the coupling ratios are determined the higher-order elasticity follows. We will see that the three-point couplings (for which a "fusing rule" was proposed in [3]) play an important role in this.

In the following we give a very brief description of affine Toda field theory. Affine Toda field theory[1] [5] is a massive scalar field theory with exponential interactions in $1+1$ dimensions described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi} - \frac{m^2}{\beta^2}\sum_{i=0}^{r} n_i \mathrm{e}^{\beta\boldsymbol{\alpha}_i\cdot\boldsymbol{\phi}}. \tag{1.1}$$

The field $\boldsymbol{\phi}$ is an r-component scalar field, r is the rank of a compact semi-simple Lie algebra G. $\boldsymbol{\alpha}_i$, $i=1,\dots,r$, are simple roots and $\boldsymbol{\alpha}_0$ is the affine root of G. The roots are normalized so that long roots have length $\sqrt{2}$, i.e. $\boldsymbol{\alpha}_{\mathrm{L}}^2=2$. The Kac–Coxeter labels n_i are such that $\sum_{i=0}^{r} n_i\boldsymbol{\alpha}_i = 0$, with the convention $n_0=1$. The quantity $\sum_{i=0}^{r} n_i$ is denoted by "h" and is known as the Coxeter number. "m" is a real parameter setting the mass scale of the theory and β is a real coupling constant, which is relevant only in the quantum theory.

ATFT is the best theoretical laboratory for understanding quantum field theory "beyond perturbation". ATFT with real coupling is one of the best understood field theories at the classical and quantum levels. ATFT is integrable at the classical level [5,6] due to the presence of an infinite number of conserved quantities. Based on the assumption that the infinite set of conserved quantities be preserved after quantization, only the elastic processes are allowed and the multi-particle S-matrices are factorized into a product of two particle elastic S-matrices [7]. In ATFT, it is well known that these conserved quantities are related with the Cartan matrix of the associated finite Lie algebra. Higher-spin quantum conserved currents are discussed in [8]. Exact quantum S-matrices for all simply laced ATFT were evaluated in [9–14]. Most of the non-simply laced ATFT exact S-matrices were calculated in [15] with the beautiful idea of floating masses. These S-matrices respect crossing symmetry and the bootstrap principle [7,11]. The exact quantum S-matrices for the remaining non-simply laced theory were constructed in [16], where the generalized bootstrap principle was introduced and more insight in the mechanism was provided. The singularity structure of the S-matrices of simply laced theories, which in some cases contain poles up to 12th order [11], is beautifully explained in terms of the singularities of the corresponding Feynman diagrams [17], the so-called Landau singularities. Finally, affine Toda field theory is a place where one can see explicitly the recently popular strong-weak coupling duality. It is known that exact Toda S-matrices for simply laced systems are invariant under this duality.

The next section presents the results obtained in the opening section of [1]. Section 3 solves the exercise suggested at the end of the opening section of [1] to obtain the $a_2^{(2)}$ ATFT or the Bullough–Dodd theory. Section 4 works with two scalar fields of different masses (one has a mass of $\sqrt{2}$ times the other) and an interaction between them. This problem leads to an ATFT associated with the $c_2^{(1)}$

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[1] For an excellent review see [4].

root system, which is non-simply laced. Section 5 will deal with a general approach towards theories with two scalar particles which are self-conjugate. In this section the allowed values of the mass ratio and three-point couplings would be obtained and the final theory would come out to be an ATFT associated with a rank two root system. Section 6 is reserved for conclusions and a conjecture.

2 sinh-Gordon or $a_1^{(1)}$ theory

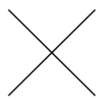
This section is shamelessly lifted from the Introduction of [1]. Starting from scalar ϕ^4 theory in $1+1$ dimensions the simplest possible ATFT i.e. sinh-Gordon or $a_1^{(1)}$ is obtained.

We begin with the $1+1$ dimensional scalar ϕ^4 Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{2.1}$$

The Feynman rules are

$$\overline{\qquad\qquad\qquad} = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon},$$



$$= -\mathrm{i}\lambda$$

,

where p is the momentum and m is the mass of the particles. We use light-cone coordinates,

$$(p, \bar{p}) = (p^0 + p^1, p^0 - p^1).$$

Using the mass-shell condition $p\bar{p} = m^2$, the *in* and *out* momenta are written as

$$(p_a, \bar{p_a}) = (ma, ma^{-1}),\ (p_b, \bar{p_b}) = (mb, mb^{-1})$$

and so on, with $a, b, \ldots$ real numbers, positive for a particle traveling forward in time. One now calculates the connected $2\phi \to 4\phi$ production amplitude at tree level. For this one looks at the diagrams of $3\phi \to 3\phi$ processes, with the implicit understanding that one of the *in* momenta will be crossed to *out* at the end. The *in* particles are labeled a, b, c and the *out* particles as d, e, f. In terms of these variables crossing from $3\phi \to 3\phi$ to $2\phi \to 4\phi$ amounts to a continuation from c to $-c$. For the $3\phi \to 3\phi$ amplitude at tree level there are just the two classes of diagrams of Fig. 1, as shown in [1].

As one of the *in* momenta is actually going *out*, the propagator is not on the mass-shell, so removal of iϵ terms is allowed. Thus the internal momentum is $p = m(a+b-d, a^{-1}+b^{-1}-d^{-1})$, and the contribution to the propagator from Fig. 1a is

$$\frac{\mathrm{i}}{p^2 - m^2} = \frac{\mathrm{i}}{m^2[(a+b-d)(a^{-1}+b^{-1}-d^{-1}) - 1]} = \frac{-\mathrm{i}abd}{m^2(a+b)(a-d)(b-d)}. \tag{2.2}$$

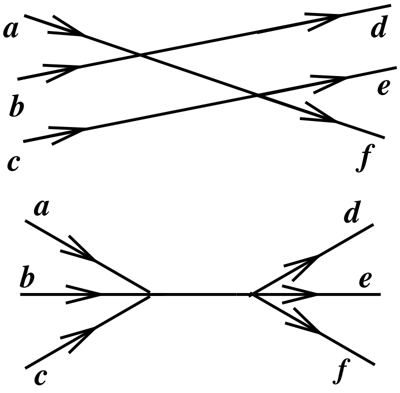


Fig. 1. $3\phi \to 3\phi$ process

Similarly for Fig. 1b the contribution to the propagator is

$$\frac{\mathrm{i}}{p^2 - m^2} = \frac{\mathrm{i}abc}{m^2(a+b)(a+c)(b+c)}. \tag{2.3}$$

Taking all the terms in account the amplitude of *in* $\to$ *out* is

$$\langle out \,|\, in \rangle_{\text{tree}} = -\frac{\mathrm{i}\lambda^2}{m^2} A_{\text{legs}} H(a,b,c,d,e,f), \tag{2.4}$$

where A_{legs} contains all the common factors on the external legs, and

$$H(a,b,c,d,e,f) = \left[\sum_{\substack{\text{cycl}\{a,b,c\}\\ \text{cycl}\{d,e,f\}}} \frac{-abd}{(a+b)(a-d)(b-d)}\right] + \frac{abc}{(a+b)(b+c)(c+a)}. \tag{2.5}$$

Using $a+b+c = d+e+f$ and $a^{-1}+b^{-1}+c^{-1} = d^{-1}+e^{-1}+f^{-1}$, i.e. the conservation of left- and right-light-cone momenta respectively, one finds $H(a,b,c,d,e,f) = -1$. As the above argument does not contain the sign of any momenta, it holds for $-c$ also.

So we find in $1+1$ dimensional $\lambda\phi^4$ theory that the amplitude of $2\phi \to 4\phi$ is constant at tree level.

By adding a term $-\frac{\lambda^2}{6!m^2}\phi^6$ to the original Lagrangian (2.1) one can make the $2\phi \to 4\phi$ amplitude vanish. Defining $\beta^2 = \lambda/m^2$, the new Lagrangian up to ϕ^6 order is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\left[\frac{1}{2}\beta^2\phi^2 + \frac{1}{4!}\beta^4\phi^4 + \frac{1}{6!}\beta^6\phi^6\right]. \tag{2.6}$$

Now one calculates the $2\phi \to 6\phi$ tree level amplitude with this $-\frac{\lambda^2}{6!m^2}\phi^6$ term added to the Lagrangian (2.1) and finds it to be a constant, which can be canceled by a judiciously chosen ϕ^8 term and so on. At each stage a residual constant piece can be removed by a (uniquely determined) higher-order interaction. After adding infinitely many terms in this way assuring no particle production at tree level one finds

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial\phi)^2 \\ &- \frac{m^2}{\beta^2}\left[\frac{1}{2!}\beta^2\phi^2 + \frac{1}{4!}\beta^4\phi^4 + \frac{1}{6!}\beta^6\phi^6 + \frac{1}{8!}\beta^8\phi^8 \dots\right] \\ &= \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\left[\cosh(\beta\phi) - 1\right] = \mathcal{L}_{a_1^{(1)}}.\end{aligned} \tag{2.7}$$

The above Lagrangian, the simplest ATFT, is the sinh-Gordon or $a_1^{(1)}$ Lagrangian and is well studied in the literature. Araf'eva and Korepin showed in [18] that the elasticity is maintained at one loop level for the above Lagrangian. The well-known sine-Gordon Lagrangian could be obtained by sending the coupling β to imaginary values.

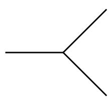
3 Bullough–Dodd model or $a_2^{(2)}$ theory

What would happen if we played the same game with a ϕ^3 theory? This was suggested as an exercise in [1]. The solution follows here in detail. The Lagrangian we begin with has the following form:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\eta}{3!}\phi^3. \tag{3.1}$$

Again the Feynman rules are

 $= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

 $= -\mathrm{i}\eta$

First we consider a $2\phi \to 3\phi$ process for which we have the tree level diagram of Fig. 2. For making our calculations easier we take both *in* momenta ($(a,\ a^{-1})$ and $(b,\ b^{-1})$) equal to (1,1) and one of the *out* momenta, (e, e^{-1}), equal to $(1+\delta, (1+\delta)^{-1})$; δ need not be small.

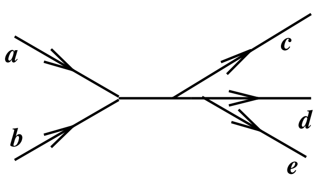


Fig. 2. $2\phi \to 3\phi$ process with three ϕ^3 vertices

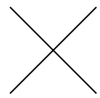
Now, the conservation of left- and right-light-cone momenta would give

$$c + d = 1 - \delta, \quad c^{-1} + d^{-1} = \frac{1+2\delta}{1+\delta}, \tag{3.2}$$
$$cd = \frac{1-\delta^2}{1+2\delta}, \quad cd^{-1} + c^{-1}d = -\frac{1+\delta+2\delta^2}{1+\delta},$$

where (c, c^{-1}) and (d, d^{-1}) are the momenta of the other two outgoing particles. There are altogether fifteen diagrams of the above type (details of their individual contributions are given in Appendix A1). Summing all the diagrams using the above relations, (3.2), we obtain

$$\begin{aligned}\frac{\mathrm{i}\eta^3}{m^4}\Bigg[&-\frac{3}{2}\frac{(1+\delta)}{\delta^2} - 1 + \frac{3}{2}\frac{(4+\delta)}{(1+\delta+\delta^2)} \\ &+ \frac{9}{2}\frac{\delta}{(1+\delta+\delta^2)^2}\Bigg].\end{aligned} \tag{3.3}$$

For stopping the particle production at tree level we add a counter term $-\dfrac{\lambda}{4!}\phi^4$ to the Lagrangian (3.1). This would produce a new Feynman rule,

 $= -\mathrm{i}\lambda,$

giving the class of new diagrams of Fig. 3 for the tree level $2\phi \to 3\phi$ process.

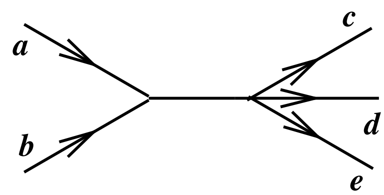
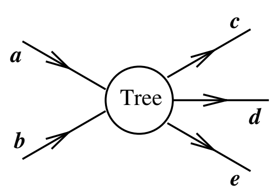


Fig. 3. $2\phi \to 3\phi$ process with a ϕ^3 vertex and a ϕ^4 vertex

There are a total of ten diagrams of the above type and we sum them again using the relations (3.2). The total contribution is (for individual details and contributions of the diagrams see Appendix A2)

$$\begin{aligned}\frac{\mathrm{i}\eta\lambda}{m^2}\Bigg[&\frac{1}{2}\frac{(1+\delta)}{\delta^2} + 2 - \frac{1}{2}\frac{(4+\delta)}{(1+\delta+\delta^2)} \\ &- \frac{3}{2}\frac{\delta}{(1+\delta+\delta^2)^2}\Bigg].\end{aligned} \tag{3.4}$$

Adding (3.4) and (3.3) we obtain the total tree level contribution of the $2\phi \to 3\phi$ process with ϕ^3 and ϕ^4 terms in the Lagrangian as



$$\begin{aligned}&= \left(\frac{\mathrm{i}2\eta\lambda}{m^2} - \frac{\mathrm{i}\eta^3}{m^4}\right) + \left(\frac{\mathrm{i}\eta\lambda}{m^2} - \frac{\mathrm{i}3\eta^3}{m^4}\right) \\ &\times\left[\frac{1}{2}\frac{(1+\delta)}{\delta^2} - \frac{1}{2}\frac{(4+\delta)}{(1+\delta+\delta^2)} - \frac{3}{2}\frac{\delta}{(1+\delta+\delta^2)^2}\right].\end{aligned} \tag{3.5}$$

Now if one chooses $\lambda = \frac{3\eta^2}{m^2}$, one gets rid of all the terms involving parameter δ and the total contribution becomes a constant equal to $\frac{i5\eta^3}{m^4}$. This constant contribution could be killed if another counter term, $-\frac{5\eta^3}{5!m^4}\phi^5$, is added to the Lagrangian (3.1). After adding ϕ^4 and ϕ^5 terms, the new Lagrangian looks like

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\eta}{3!}\phi^3 - \frac{3\eta^2}{4!m^2}\phi^4 - \frac{5\eta^3}{5!m^4}\phi^5 \cdots \quad (3.6)$$

The above Lagrangian, (3.6), would produce a vanishing result for the tree level $2\phi \to 3\phi$ process. To fix the dots in the above Lagrangian one should look for the $2\phi \to 4\phi$ tree level process. Next one demands a vanishing contribution for this $2\phi \to 4\phi$ tree level process to fix the ϕ^6 term and one can proceed thus order by order.

Setting $\beta = \frac{\eta}{m^2}$ we observe that the Lagrangian, (3.6), contains the first four terms of the following Lagrangian after a power series expansion:

$$\mathcal{L}_{\boldsymbol{a_2^{(2)}}} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{6\beta^2}\left[\, \mathrm{e}^{2\beta\phi} + 2\mathrm{e}^{-\beta\phi} - 3 \,\right]. \quad (3.7)$$

The above, (3.7), is another well studied Lagrangian known as the Bullough–Dodd model or the $a_2^{(2)}$ ATFT in the literature. We mention in passing that if one starts with a theory with both ϕ^3 and ϕ^4 terms absent in the Lagrangian, then no particle production at tree level would lead to a free theory that is having all higher couplings vanishing.

4 $c_2^{(1)}$ theory

In this section we consider two interacting self-conjugate scalar fields. The starting Lagrangian in this case is chosen to be

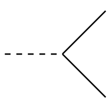
$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 + \frac{\xi}{2!}\phi_1\phi_2^2. \quad (4.1)$$

Notice that the particle ϕ_1 is $\sqrt{2}$ times heavier than the particle ϕ_2 and there is only one interaction term, viz. $\phi_1\phi_2^2$.

The Feynman rules are the following:

ϕ_1 propagator : - - - - - - - - - - $= \dfrac{\mathrm{i}}{p^2 - 2m^2 + \mathrm{i}\epsilon}$,

ϕ_2 propagator : ———————— $= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$,


$= \mathrm{i}\xi$.

First we consider the process $2\phi_2 \to 2\phi_1$. At tree level, there are the two diagrams of Fig. 4.

The *in* particles have momenta (a, a^{-1}) and (b, b^{-1}) whereas the *out* ones have momenta $(\sqrt{2}\,c, \sqrt{2}\,c^{-1})$ and $(\sqrt{2}\,d, \sqrt{2}\,d^{-1})$ without loss of generality, since the *out*

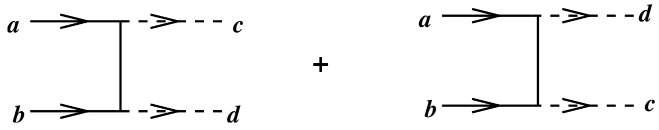


Fig. 4. $2\phi_2 \to 2\phi_1$ process

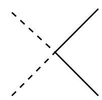
particles are $\sqrt{2}$ times as heavy as the *in* ones. The conservation of left- and right-light-cone momenta gives the following relations:

$$c + d = \frac{a+b}{\sqrt{2}}, \quad c^{-1} + d^{-1} = \frac{a^{-1}+b^{-1}}{\sqrt{2}}, \quad (4.2)$$
$$cd = ab, \quad cd^{-1} + c^{-1}d + 1 = \frac{1}{2}(ab^{-1} + a^{-1}b).$$

The individual contributions of the above two diagrams are

$$\frac{-\mathrm{i}\xi^2}{m^2[2 - \sqrt{2}(ac^{-1} + a^{-1}c)] + \mathrm{i}\epsilon} + \frac{-\mathrm{i}\xi^2}{m^2[2 - \sqrt{2}(ad^{-1} + a^{-1}d)] + \mathrm{i}\epsilon}. \quad (4.3)$$

Summing the above two expressions using the relations (4.2) and then taking the limit $\epsilon \to 0$ we obtain a constant equal to $\frac{\mathrm{i}\xi^2}{m^2}$. This constant contribution will be killed if we add a term, $-\frac{1}{2!2!}\frac{\xi^2}{m^2}\phi_1^2\phi_2^2$, to the Lagrangian (4.1). This new term gives an additional Feynman rule,


$= -\dfrac{\mathrm{i}\xi^2}{m^2}$.

Next we look at the process $2\phi_2 \to 2\phi_2 + \phi_1$. Again for simplifying our calculations we take the *in* momenta as (1,1) and the momentum of the outgoing particle ϕ_1 as $(\sqrt{2}e, \sqrt{2}e^{-1})$. We have the following two types of diagrams (Fig. 5): one having three $\phi_1\phi_2^2$ vertices and the other containing one $\phi_1\phi_2^2$ vertex and one $\phi_1^2\phi_2^2$ vertex.

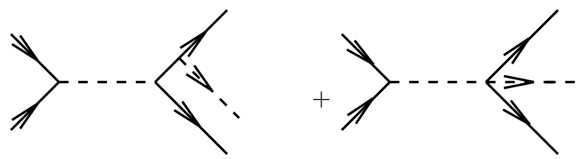


Fig. 5. $2\phi_2 \to 2\phi_2 + \phi_1$ process (a) with three $\phi_1\phi_2^2$ vertices and (b) with one $\phi_1\phi_2^2$ vertex and one $\phi_1^2\phi_2^2$ vertex

We have twelve diagrams of the former type and six diagrams of the latter type (details of which are presented in the Appendix B1). Summing all 18 diagrams one obtains (using momentum conservation relations of course)

$$-\frac{\mathrm{i}\xi^3}{m^4}\frac{e\,(8e - 3\sqrt{2}\,e^2 - 3\sqrt{2})}{(2e - \sqrt{2}\,e^2 - \sqrt{2})^2}. \quad (4.4)$$

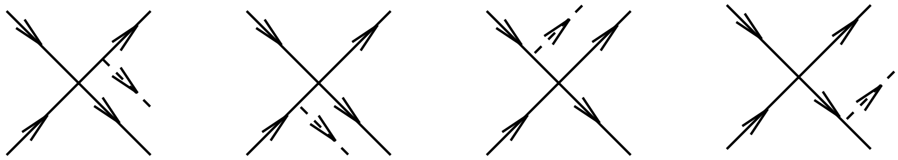


Fig. 6. $2\phi_2 \to 2\phi_2 + \phi_1$ with a ϕ_2^4 vertex and a $\phi_1\phi_2^2$ vertex

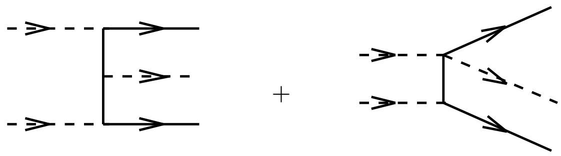
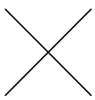


Fig. 7. $2\phi_1 \to 2\phi_2 + \phi_1$ process

Now if one adds a counter term $-\frac{\zeta}{4!}\phi_2^4$ to the Lagrangian, (4.1) one will have a new vertex of the following type:

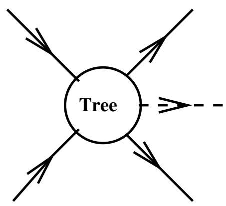


$$= -\mathrm{i}\zeta$$

This new vertex in turn would add the four more diagrams of Fig. 6 to the above process $2\phi_2 \to 2\phi_2 + \phi_1$. The contribution of these four diagrams when added is equal to

$$-\frac{\mathrm{i}\xi\zeta}{m^2} + \frac{\mathrm{i}\xi\zeta}{m^2}\frac{e\,(8e - 3\sqrt{2}\,e^2 - 3\sqrt{2})}{(2e - \sqrt{2}\,e^2 - \sqrt{2})^2}. \tag{4.5}$$

Adding (4.4) and (4.5) we have



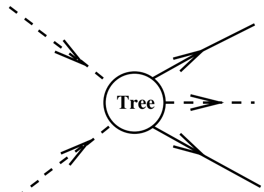
$$= -\frac{\mathrm{i}\xi\zeta}{m^2} + \left(\frac{\mathrm{i}\xi\zeta}{m^2} - \frac{\mathrm{i}\xi^3}{m^4}\right)\frac{e\,(8e - 3\sqrt{2}\,e^2 - 3\sqrt{2})}{(2e - \sqrt{2}\,e^2 - \sqrt{2})^2} \tag{4.6}$$

The above expression, (4.6), clearly shows if we choose $\zeta = \frac{\xi^2}{m^2}$ then the above expression becomes independent of the parameter e and the total sum becomes $-\frac{\mathrm{i}\xi^3}{m^4}$. This constant contribution will be killed if one adds a new counter term $\frac{1}{4!}\frac{\xi^3}{m^4}\phi_1\phi_2^4$ to the Lagrangian (4.1). At this stage our Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 + \frac{\xi}{2!}\phi_1\phi_2^2 - \frac{1}{2!2!}\frac{\xi^2}{m^2}\phi_1^2\phi_2^2 - \frac{1}{4!}\frac{\xi^2}{m^2}\phi_2^4 + \frac{1}{4!}\frac{\xi^3}{m^4}\phi_1\phi_2^4. \tag{4.7}$$

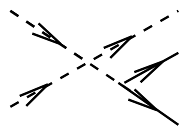
To fix the remaining quartic and quintic interactions we concentrate on the $2\phi_1 \to 2\phi_2 + \phi_1$ process which possesses the two classes of diagrams of Fig. 7, viz. one with three $\phi_1\phi_2^2$ vertices and the other containing one $\phi_1\phi_2^2$ vertex and one $\phi_1^2\phi_2^2$ vertex like before.

In this case we choose the *in* momenta for both particles as $(\sqrt{2}, \sqrt{2})$ and the momentum of the outgoing particle ϕ_1 is designated by $(\sqrt{2}\,e, \sqrt{2}\,e^{-1})$. There are six diagrams of each kind (see Appendix B2 for details). Summing all twelve diagrams we obtain



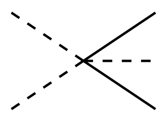
$$= -\frac{\mathrm{i}\xi^3}{m^4} + \frac{\mathrm{i}\xi^3}{2m^4}\frac{e}{(e-1)^2}. \tag{4.8}$$

From the above expression (4.8) it is easy to fix the quartic ϕ_1^4 counter term so that the parameter e dependent term is killed. For this we add a term $-\frac{\gamma}{4!}\phi_1^4$ to the Lagrangian (4.7) to obtain the following new diagram for the above process:



$$= -\frac{\mathrm{i}\xi\gamma}{4m^2}\frac{e}{(e-1)^2}. \tag{4.9}$$

Now it is very clear from the above expressions (4.8) and (4.9), if we choose $\gamma = \frac{2\xi^2}{m^2}$ all e dependent terms would cancel from the tree level $2\phi_1 \to 2\phi_2 + \phi_1$ process and the result would be a constant equal to $-\frac{\mathrm{i}\xi^3}{m^4}$. This constant contribution is canceled by a vertex of the following type:



$$= \frac{\mathrm{i}\xi^3}{m^4}\ . \tag{4.10}$$

The above vertex corresponds to adding a $\frac{1}{3!2!}\frac{\xi^3}{m^4}\phi_1^3\phi_2^2$ term to the Lagrangian (4.7). The final Lagrangian with all five-point interaction vertices becomes

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 + \frac{\xi}{2!}\phi_1\phi_2^2 - \frac{2}{4!}\frac{\xi^2}{m^2}\phi_1^4 - \frac{1}{2!2!}\frac{\xi^2}{m^2}\phi_1^2\phi_2^2 - \frac{1}{4!}\frac{\xi^2}{m^2}\phi_2^4 + \frac{1}{4!}\frac{\xi^3}{m^4}\phi_1\phi_2^4 + \frac{1}{3!2!}\frac{\xi^3}{m^4}\phi_1^3\phi_2^2\ . \tag{4.11}$$

The above Lagrangian, (4.11), contains the first eight terms (after expansion) of the following $c_2^{(1)}$ ATFT Lagrangian:

$$\mathcal{L}_{\mathbf{c_2^{(1)}}} = \frac{1}{2}\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi} - \frac{m^2}{2\beta^2}\left[\mathrm{e}^{\beta\boldsymbol{\alpha}_0\cdot\boldsymbol{\phi}} + 2\mathrm{e}^{\beta\boldsymbol{\alpha}_1\cdot\boldsymbol{\phi}} + \mathrm{e}^{\beta\boldsymbol{\alpha}_2\cdot\boldsymbol{\phi}} - 4\right], \tag{4.12}$$

where the field $\boldsymbol{\phi}$ has two components i.e. ϕ_1 and ϕ_2. $\boldsymbol{\alpha}_0$ is affine and $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are simple roots of the algebra $c_2^{(1)}$ ($\boldsymbol{\alpha}_1 = (1,0)$, $\boldsymbol{\alpha}_2 = (-1,1)$, $\boldsymbol{\alpha}_0 = (-1,-1)$) and $\beta = \frac{\xi}{m^2}$. This is another integrable model which is well studied [11]. Again all the higher n-point couplings ($n > 5$) could be fixed by studying the various other tree level processes.

5 Other theories with two self-conjugate scalar fields

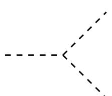
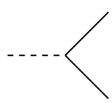
In this section we give a general method for constructing various other integrable theories associated with rank two root systems. We have seen in the previous section that the sole three-point interaction decides the fate of the other terms if one maintains the elasticity property order by order at tree level. One can verify that elasticity is maintained if one goes to loop diagrams. Here we start with the most general Lagrangian with two self-conjugate scalar fields with all possible three-point interactions,

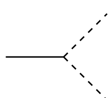
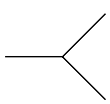
$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - \frac{q}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 + \frac{\xi}{2!}\phi_1\phi_2^2 - \frac{r\xi}{3!}\phi_1^3 - \frac{s\xi}{3!}\phi_2^3 - \frac{t\xi}{2!}\phi_1^2\phi_2. \tag{5.1}$$

Note that we have fixed the strength of one mass term and only one of the three-point interactions. The other mass and couplings have strengths relative to these. Our objective is to determine these relative strengths (i.e. q, r, s and t) for an integrable theory. The Feynman rules are given by

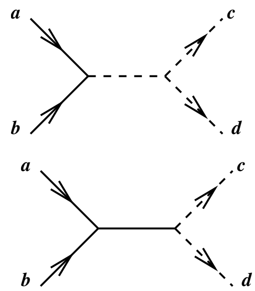
ϕ_1 propagator : $= \dfrac{\mathrm{i}}{p^2 - q\,m^2 + \mathrm{i}\epsilon}$,

ϕ_2 propagator : $= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$,

 $= -\mathrm{i}r\xi$,  $= \mathrm{i}\xi$,

 $= -\mathrm{i}t\xi$,  $= -\mathrm{i}s\xi$.

We start with the process $2\phi_2 \to 2\phi_1$, as in the previous section and calculate all possible tree level diagrams with the above Lagrangian. For some particular combinations of q, r, s and t only the contribution of the $2\phi_2 \to 2\phi_1$ process turns out to be a constant, i.e. independent of the *in* momenta, and in that case this constant contribution can be killed by adding a judiciously chosen $\phi_1^2\phi_2^2$ term to the above Lagrangian, (5.1). In this way one decides all possible three-point functions for a particular theory to be constructed. Next one proceeds in the manner explained in the previous section, viz. studying the other tree level processes and fixing the higher-order interaction terms. Each of these combination of three-point functions (i.e. combination of q, r, s and t) gives an integrable model associated with a rank two root system. In this section we would only fix the three-point couplings by studying the $2\phi_2 \to 2\phi_1$ process in detail. In the following are the six diagrams (Figs. 8, 9, 10) contributing to the process $2\phi_2 \to 2\phi_1$, where $x \equiv ab^{-1} + a^{-1}b$. Using conservation of



$$= \frac{\mathrm{i}\xi^2}{m^2}\frac{r}{(2-q+x)}\,,$$

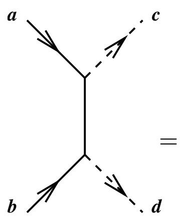
$$= -\frac{\mathrm{i}\xi^2}{m^2}\frac{st}{(1+x)},$$

Fig. 8. The $2\phi_2 \to 2\phi_1$ process

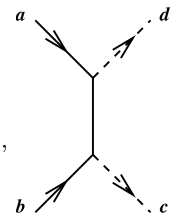
left- and right-light-cone momenta,

$$\sqrt{q}\,(c+d) = a+b \quad\text{and}\quad \sqrt{q}\,(c^{-1}+d^{-1}) = a^{-1}+b^{-1}, \tag{5.2}$$

respectively, one obtains $x = q(cd^{-1} + c^{-1}d) + 2(q-1)$.

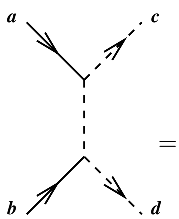


$$= -\frac{\mathrm{i}\xi^2}{m^2}\frac{1}{(q - \sqrt{q}\,(ac^{-1} + a^{-1}c))}\,,$$

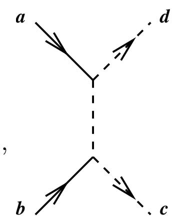


$$= -\frac{\mathrm{i}\xi^2}{m^2}\frac{1}{(q - \sqrt{q}\,(ad^{-1} + a^{-1}d))}\,.$$

Fig. 9. $2\phi_2 \to 2\phi_1$ process with two $\phi_1\phi_2^2$ vertices



$$= -\frac{\mathrm{i}\xi^2}{m^2}\frac{t^2}{(1 - \sqrt{q}\,(ac^{-1} + a^{-1}c))}\,,$$



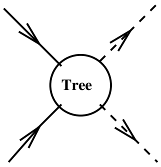
$$= -\frac{\mathrm{i}\xi^2}{m^2}\frac{t^2}{(1 - \sqrt{q}\,(ad^{-1} + a^{-1}d))}\,.$$

Fig. 10. $2\phi_2 \to 2\phi_1$ process with two $\phi_1^2\phi_2$ vertices

Summing both diagrams of Fig. 9, we get

$$\frac{\mathrm{i}\xi^2}{m^2}\frac{(2-2q+x)}{(q^2-4q+2+x)},$$

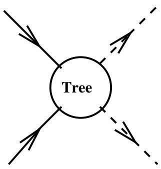
using (5.2). Adding both the diagrams of Fig. 10, we obtain $\frac{\mathrm{i}\xi^2}{m^2}\frac{t^2x}{(1-2q+qx)}$, using (5.2) again. The total contribution of all the six diagrams then becomes



$$= \frac{\mathrm{i}\xi^2}{m^2}\left[\frac{r}{(2-q+x)} - \frac{st}{(1+x)} + \frac{(2-2q+x)}{(q^2-4q+2+x)} + \frac{t^2x}{(1-2q+qx)}\right]. \tag{5.3}$$

Now one looks for cases for which expression (5.3) is a constant, i.e. independent of x (or incoming momenta), so that it could be killed by adding a $\phi_1^2\phi_2^2$ term to the Lagrangian (5.1) with a suitably chosen coefficient.

Case I. $t = 0$[2]. This gives a contribution (from the right hand side of (5.3)),



$$= \frac{\mathrm{i}\xi^2}{m^2}\left[\frac{r}{(2-q+x)} + \frac{(2-2q+x)}{(q^2-4q+2+x)}\right] = \frac{\mathrm{i}\xi^2}{m^2} \times \left[\frac{x^2 + x(4-3q+r) + (2q^2-6q+4+r(q^2-4q+2))}{x^2+x(q^2-5q+4)+(-q^3+6q^2-10q+4)}\right]. \tag{5.4}$$

To have a constant contribution one must now equate the coefficients of various powers of x in the numerator and denominator within the square brackets of the expression (5.4). In this case we get two equations. Equating coefficients of x we get

$$q^2 - 2q = r, \tag{5.5}$$

and equating the constants we have

$$q^3 + q^2(r-4) + 4q(1-r) + 2r = 0. \tag{5.6}$$

Using (5.5) in (5.6), we get three solutions, viz. $(q = 0, r = 0)$; $(q = 2, r = 0)$ and $(q = 3, r = 3)$.

(a) The first of these, viz. $(q = 0, r = 0)$ is not acceptable as it sets the mass of particle ϕ_1 to zero.

(b) The second solution $(q = 2, r = 0)$ is already discussed in detail in the last section and leads to $\boldsymbol{c_2^{(1)}}$ ATFT. Moreover one has to choose $s = 0$ for that. It is clear that s cannot be determined from the above as it vanishes from the expression once one chooses $t = 0$ in (5.3). Of course it can be fixed demanding a zero contribution from the other tree level processes.

(c) The third solution $(q = 3, r = 3)$ will lead to two different theories depending on the values chosen for s. The allowed values of s can be again fixed by studying other tree level processes.

(i) If $s = 0$, then the theory is $\boldsymbol{d_3^{(2)}}$ with the Lagrangian

$$\mathcal{L}_{\boldsymbol{d_3^{(2)}}} = \frac{1}{2}\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi} - \frac{m^2}{\beta^2}\left[\sum_{i=0}^{2}\mathrm{e}^{\beta\boldsymbol{\alpha}_i\cdot\boldsymbol{\phi}} - 3\right], \tag{5.7}$$

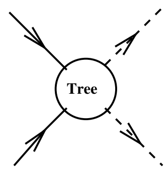
with simple and affine roots $\boldsymbol{\alpha}_1 = (\sqrt{2}, 0)$, $\boldsymbol{\alpha}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\boldsymbol{\alpha}_0 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\beta = \sqrt{2}\,\xi/m^2$.

(ii) If $s = -\frac{2}{\sqrt{3}}$, then the theory is $\boldsymbol{g_2^{(1)}}$ and the corresponding Lagrangian becomes

$$\mathcal{L}_{\boldsymbol{g_2^{(1)}}} = \frac{1}{2}\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi} - \frac{m^2}{2\beta^2}\left[\mathrm{e}^{\beta\boldsymbol{\alpha}_0\cdot\boldsymbol{\phi}} + 3\mathrm{e}^{\beta\boldsymbol{\alpha}_1\cdot\boldsymbol{\phi}} + 2\mathrm{e}^{\beta\boldsymbol{\alpha}_2\cdot\boldsymbol{\phi}} - 6\right], \tag{5.8}$$

with simple and affine roots $\boldsymbol{\alpha}_1 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right)$, $\boldsymbol{\alpha}_2 = (\sqrt{2}, 0)$, $\boldsymbol{\alpha}_0 = \left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$ and $\beta = \sqrt{2}\,\xi/m^2$.

Case II. $r = s = 0$. In this case again we demand a contribution (from the right hand side of (5.3))



$$= \frac{\mathrm{i}\xi^2}{m^2}\left[\frac{(2-2q+x)}{(q^2-4q+2+x)} + \frac{t^2x}{(1-2q+qx)}\right] = \frac{\mathrm{i}\xi^2}{m^2} \times \left[\frac{\left(1+\frac{t^2}{q}\right)x^2 + \left(t^2q - 4t^2 - 2q + \frac{1}{q} + \frac{2t^2}{q}\right)x + (2-2q)\left(\frac{1}{q}-2\right)}{x^2 + \left(q^2-4q+\frac{1}{q}\right)x + (q^2-4q+2)\left(\frac{1}{q}-2\right)}\right], \tag{5.9}$$

for the process $2\phi_2 \to 2\phi_1$ to be a constant. Proceeding in exactly the previous way (i.e. matching the coefficients of various powers x in numerator and denominator) we have the following two distinct solutions, viz. $(q = \frac{3+\sqrt{5}}{2}, t = \frac{1+\sqrt{5}}{2})$ and $(q = 1, t^2 = -1)$.

(a) The first solution, $(q = \frac{3+\sqrt{5}}{2}, t = \frac{1+\sqrt{5}}{2})$, leads to the $\boldsymbol{a_4^{(2)}}$ ATFT, the Lagrangian for which is given by

$$\mathcal{L}_{\boldsymbol{a_4^{(2)}}} = \frac{1}{2}\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi} - \frac{2m^2}{5\beta^2}\frac{1}{(1-\sin 2\theta)} \times \left[\mathrm{e}^{\beta\boldsymbol{\alpha}_0\cdot\boldsymbol{\phi}} + 2e^{\beta\boldsymbol{\alpha}_1\cdot\boldsymbol{\phi}} + 2e^{\beta\boldsymbol{\alpha}_2\cdot\boldsymbol{\phi}} - 5\right], \tag{5.10}$$

where the affine and simple roots are

$\boldsymbol{\alpha}_0 = (-\sqrt{2}\sin(\pi/4+\theta), \sqrt{2}\cos(\pi/4+\theta))$,
$\boldsymbol{\alpha}_1 = (\cos\theta, \sin\theta)$,
$\boldsymbol{\alpha}_2 = \left(-\frac{1}{\sqrt{2}}\sin(\pi/4-\theta), -\frac{1}{\sqrt{2}}\cos(\pi/4-\theta)\right)$,
with $2\tan 2\theta = 1$ and

[2] One need not fix $t = 0$ (or $r = s = 0$ as done in case II discussed later) a priori. One could study the entire expression (5.3) as will be done in case III later. The constancy constraint on (5.3) would produce cases I and II as solutions.

$$\beta = \frac{(1-\csc 2\theta)}{(\sin\theta - \cos\theta)}\frac{\xi}{m^2}.$$

(b) It is clear from the expression (5.9) that the second solution ($q = 1, t^2 = -1$) will result in a vanishing contribution for the above process. This also asks for an imaginary coupling t. This, we believe, should lead to the $\boldsymbol{a_2^{(1)}}$ ATFT, which is another rank two ATFT available with mass ratio of two fields of value unity (i.e. $q = 1$). One must note that in $\boldsymbol{a_2^{(1)}}$ theory the two fields are not self-conjugate but mutually conjugate.

Case III. $r \neq 0, s \neq 0, t \neq 0$. In this case we obtain four equations (equating various powers of x in the numerator and the denominator of the expression (5.3), as done earlier) by demanding the contribution of the same $2\phi_2 \to 2\phi_1$ process to be a constant. After some cumbersome algebra we reach the following solution: ($q = 2+\sqrt{3},\ r = -3-2\sqrt{3},\ t = 2+\sqrt{3},\ s = -\sqrt{3}$). This leads to the last remaining rank two ATFT, viz. the theory associated with the $\boldsymbol{d_4^{(3)}}$ root system. Lagrangian for this is

$$\mathcal{L}_{\boldsymbol{d_4^{(3)}}} = \frac{1}{2}\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi} \tag{5.11}$$
$$- \frac{\sqrt{3}m^2}{2(\sqrt{3}-1)\beta^2}\left[\mathrm{e}^{\beta\boldsymbol{\alpha}_0\cdot\boldsymbol{\phi}} + \mathrm{e}^{\beta\boldsymbol{\alpha}_1\cdot\boldsymbol{\phi}} + 2\mathrm{e}^{\beta\boldsymbol{\alpha}_2\cdot\boldsymbol{\phi}} - 4\right],$$

where the simple and affine roots are
$\boldsymbol{\alpha}_1 = \left(\frac{1+\sqrt{3}}{2},\ \frac{1-\sqrt{3}}{2}\right)$, $\boldsymbol{\alpha}_2 = \left(-\frac{1+\sqrt{3}}{2\sqrt{3}},\ \frac{1-\sqrt{3}}{2\sqrt{3}}\right)$,
$\boldsymbol{\alpha}_0 = \left(\frac{\sqrt{3}-1}{2\sqrt{3}},\ \frac{\sqrt{3}+1}{2\sqrt{3}}\right)$ and $\beta = -2\sqrt{3}\,\xi/m^2$.

This completes our list of distinct solutions. There are other solutions like

$$\left(q = \frac{3+\sqrt{5}}{2}, t = -\frac{1+\sqrt{5}}{2}\right), \left(q = \frac{3-\sqrt{5}}{2}, t = \frac{1-\sqrt{5}}{2}\right)$$

and

$$(q = 2-\sqrt{3},\ r = -3+2\sqrt{3},\ t = 2-\sqrt{3},\ s = \sqrt{3})$$

etc. which would keep expression (5.3) constant but these are not distinct in the sense that they would not produce any new ATFT. The first one could be obtained from case II (a) by changing the field ϕ_2 to $-\phi_2$. The second of the above is again the same as the solution II (a). In this case the roles of the fields ϕ_1 and ϕ_2 are interchanged. Both of these would lead to the same $\boldsymbol{a_4^{(2)}}$ theory. The last solution viz. ($q = 2-\sqrt{3},\ r = -3+2\sqrt{3},\ t = 2-\sqrt{3},\ s = \sqrt{3}$) is again the same as case III with $\phi_1 \leftrightarrow \phi_2$ and leads to the $\boldsymbol{d_4^{(3)}}$ ATFT.

The above cases exhaust all possible solutions or acceptable values of q, r, s and t which will respect elasticity and also exhaust all possible ATFT associated with rank two root systems, viz. $\boldsymbol{a_2^{(1)}}$, $\boldsymbol{a_4^{(2)}}$, $\boldsymbol{c_2^{(1)}}$, $\boldsymbol{d_3^{(2)}}$, $\boldsymbol{d_4^{(3)}}$ and $\boldsymbol{g_2^{(1)}}$. S-matrices and other details about these models can be found in [11–16].

Our Lagrangians for various ATFT may look a little different from the ones existing in the literature. This is due to the fact that in the expressions of these Lagrangians we have chosen the simple and affine roots in such a way that the mass matrix becomes diagonal.

Table 1. Relative strengths of the mass terms and the three-point couplings for rank two ATFT

Case	Mass terms		Three-point interaction terms				Theory
	ϕ_1^2	ϕ_2^2	ϕ_1^3	ϕ_2^3	$\phi_1^2\phi_2$	$\phi_1\phi_2^2$	
	q	–	r	s	t	–	
I b	2	1	0	0	0	-1	$\boldsymbol{c_2^{(1)}}$
I c i	3	1	3	0	0	-1	$\boldsymbol{d_3^{(2)}}$
I c ii	3	1	3	$-\frac{2}{\sqrt{3}}$	0	-1	$\boldsymbol{g_2^{(1)}}$
II a	$\frac{3+\sqrt{5}}{2}$	1	0	0	$\frac{1+\sqrt{5}}{2}$	-1	$\boldsymbol{a_4^{(2)}}$
II b	1	1	0	0	$\pm\mathrm{i}$	-1	$\boldsymbol{a_2^{(1)}}$
III	$2+\sqrt{3}$	1	$-3-2\sqrt{3}$	$-\sqrt{3}$	$2+\sqrt{3}$	-1	$\boldsymbol{d_4^{(3)}}$

6 Summary and results

Here we summarize the results. In a pedagogical way we have introduced the way of constructing integrable models in 1 + 1 dimensions. Starting with simple scalar field theories we have exploited the elasticity property (no particle production) at tree level in the scattering of scalar particles for constructing affine Toda field theory associated with rank one and rank two root systems. It has been shown that the relative masses and three-point couplings could be fixed by the vanishing amplitude of the four-point function ($2\phi_2 \to 2\phi_1$) in the case of two scalar fields. We summarize the findings of Sect. 5 in Table 1.

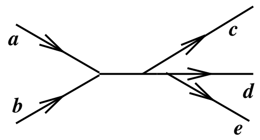
Further it was shown that once the three-point coupling are fixed, the higher-order couplings are determined uniquely by demanding the vanishing of various other scattering processes at tree level[3]. We have calculated five-point functions and verified explicitly no particle production (i.e. vanishing amplitudes for 2 *particle* → 3 *particle* processes) in $a_2^{(2)}$ and $c_2^{(1)}$ ATFT for the very first time, we believe. Each combination of allowed three-point couplings produces an ATFT associated with a particular rank two root system. We strongly believe that the same procedure could also be used for constructing ATFT associated with root systems having rank greater than two. It would be nice if one could develop a way which works for affine Toda field theories in general. Our effort is just a modest beginning in this direction.

Acknowledgements. The author thanks Prof. Ryu Sasaki for reading the manuscript and making valuable suggestions and comments at various stages of this work. The author also thanks Mr. Tapan Naskar who tried evaluating some of the Feynman diagrams at an earlier stage of this work.

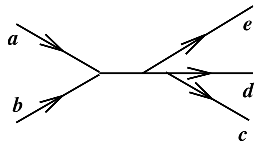
[3] One can calculate higher-order couplings from three-point couplings; see [17].

Appendix A1

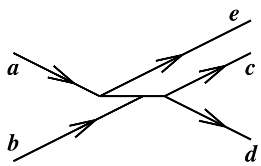
We have

1) 

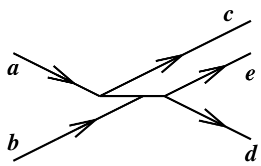
$$= -\frac{i\eta^3}{6m^4}\frac{1}{(2-c-c^{-1})} \tag{A1.1}$$

2) 

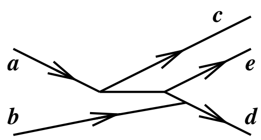
$$= \frac{i\eta^3}{6m^4}\frac{(1+\delta)}{\delta^2} \tag{A1.2}$$

3) 

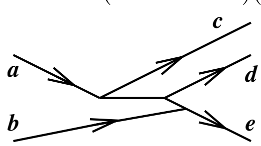
$$= -\frac{i\eta^3}{2m^4}\frac{(1+\delta)^2}{\delta^2(1+\delta+\delta^2)} \tag{A1.3}$$

4) 

$$= -\frac{i\eta^3}{2m^4}\frac{1}{(1-c-c^{-1})(2-c-c^{-1})} \tag{A1.4}$$

5) 

$$= -\frac{i\eta^3}{m^4}\frac{1}{(1-c-c^{-1})(1-d-d^{-1})} \tag{A1.5}$$

6) 

$$= \frac{i\eta^3}{m^4}\frac{1}{(1-c-c^{-1})}\frac{(1+\delta)}{(1+\delta+\delta^2)} \tag{A1.6}$$

7) $= 4)\quad (c \longleftrightarrow d)$

$$= -\frac{i\eta^3}{2m^4}\frac{1}{(1-d-d^{-1})(2-d-d^{-1})} \tag{A1.7}$$

8) $= 5)\quad (c \longleftrightarrow d)$

$$= -\frac{i\eta^3}{m^4}\frac{1}{(1-c-c^{-1})(1-d-d^{-1})} \tag{A1.8}$$

9) $= 6)\quad (c \longleftrightarrow d)$

$$= \frac{i\eta^3}{m^4}\frac{1}{(1-d-d^{-1})}\frac{(1+\delta)}{(1+\delta+\delta^2)} \tag{A1.9}$$

10) $= 1)\quad (c \longleftrightarrow d)$

$$= -\frac{i\eta^3}{6m^4}\frac{1}{(2-d-d^{-1})} \tag{A1.10}$$

11) $= 6)\quad (a \longleftrightarrow b)$

$$= \frac{i\eta^3}{m^4}\frac{1}{(1-c-c^{-1})}\frac{(1+\delta)}{(1+\delta+\delta^2)} \tag{A1.11}$$

12) $= 9)\quad (a \longleftrightarrow b)$

$$= \frac{i\eta^3}{m^4}\frac{1}{(1-d-d^{-1})}\frac{(1+\delta)}{(1+\delta+\delta^2)} \tag{A1.12}$$

13) $= 3)\quad (a \longleftrightarrow b)$

$$= -\frac{i\eta^3}{2m^4}\frac{(1+\delta)^2}{\delta^2(1+\delta+\delta^2)} \tag{A1.13}$$

14) $= 4)\quad (a \longleftrightarrow b)$

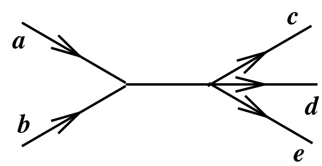
$$= -\frac{i\eta^3}{2m^4}\frac{1}{(1-c-c^{-1})(2-c-c^{-1})} \tag{A1.14}$$

15) $= 14)\quad (c \longleftrightarrow d)$

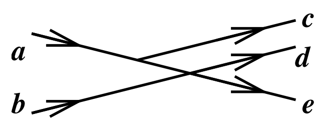
$$= -\frac{i\eta^3}{2m^4}\frac{1}{(1-d-d^{-1})(2-d-d^{-1})} \tag{A1.15}$$

Appendix A2

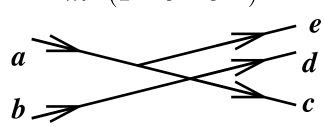
We have

1) 

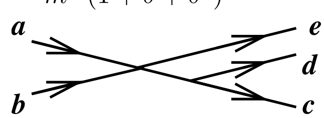
$$= -\frac{i\eta\lambda}{3m^2} \tag{A2.1}$$

2) 

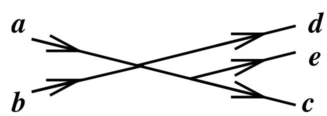
$$= -\frac{i\eta\lambda}{m^2}\frac{1}{(1-c-c^{-1})} \tag{A2.2}$$

3) 

$$= \frac{i\eta\lambda}{m^2}\frac{(1+\delta)}{(1+\delta+\delta^2)} \tag{A2.3}$$

4) 

$$= \frac{i\eta\lambda}{2m^2}\frac{(1+\delta)}{\delta^2} \tag{A2.4}$$

5) 

$$= -\frac{i\eta\lambda}{2m^2}\frac{1}{(2-d-d^{-1})} \tag{A2.5}$$

6) $= 2)\quad (a \longleftrightarrow b)$

$$= -\frac{\mathrm{i}\eta\lambda}{m^2}\frac{1}{(1-c-c^{-1})} \tag{A2.6}$$

7) $= 2)\quad (c \longleftrightarrow d)$

$$= -\frac{\mathrm{i}\eta\lambda}{m^2}\frac{1}{(1-d-d^{-1})} \tag{A2.7}$$

8) $= 7)\quad (a \longleftrightarrow b)$

$$= -\frac{\mathrm{i}\eta\lambda}{m^2}\frac{1}{(1-d-d^{-1})} \tag{A2.8}$$

9) $= 3)\quad (a \longleftrightarrow b)$

$$= \frac{\mathrm{i}\eta\lambda}{m^2}\frac{(1+\delta)}{(1+\delta+\delta^2)} \tag{A2.9}$$

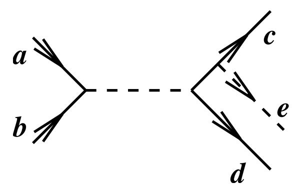
10) $= 5)\quad (c \longleftrightarrow d)$

$$= -\frac{\mathrm{i}\eta\lambda}{2m^2}\frac{1}{(2-c-c^{-1})} \tag{A2.10}$$

Appendix B1

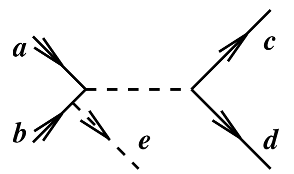
We have

1)



$$= \frac{\mathrm{i}\xi^3}{4m^4}\frac{1}{(2-d-d^{-1})} \tag{B1.1}$$

2)



$$= \frac{\mathrm{i}\xi^3}{4m^4}\frac{e^2}{(1-\sqrt{2}\,e+e^2)^2} \tag{B1.2}$$

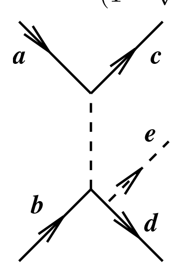
3) $= 1)\quad (c \longleftrightarrow d)$

$$= \frac{\mathrm{i}\xi^3}{4m^4}\frac{1}{(2-c-c^{-1})} \tag{B1.3}$$

4) $= 2)\quad (a \longleftrightarrow b)$

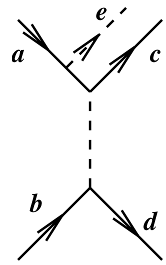
$$= \frac{\mathrm{i}\xi^3}{4m^4}\frac{e^2}{(1-\sqrt{2}\,e+e^2)^2} \tag{B1.4}$$

5)



$$= -\frac{\mathrm{i}\xi^3}{2m^4}\frac{1}{(c+c^{-1})(2-c-c^{-1})} \tag{B1.5}$$

6)



$$= \frac{\mathrm{i}\xi^3}{\sqrt{2}\,m^4}\frac{1}{(d+d^{-1})(e+e^{-1}-\sqrt{2})} \tag{B1.6}$$

7) $= 5)\quad (a \longleftrightarrow b)$

$$= -\frac{\mathrm{i}\xi^3}{2m^4}\frac{1}{(c+c^{-1})(2-c-c^{-1})} \tag{B1.7}$$

8) $= 5)\quad (c \longleftrightarrow d)$

$$= -\frac{\mathrm{i}\xi^3}{2m^4}\frac{1}{(d+d^{-1})(2-d-d^{-1})} \tag{B1.8}$$

9) $= 7)\quad (c \longleftrightarrow d)$

$$= -\frac{\mathrm{i}\xi^3}{2m^4}\frac{1}{(d+d^{-1})(2-d-d^{-1})} \tag{B1.9}$$

10) $= 6)\quad (a \longleftrightarrow b)$

$$= \frac{\mathrm{i}\xi^3}{\sqrt{2}\,m^4}\frac{1}{(d+d^{-1})(e+e^{-1}-\sqrt{2})} \tag{B1.10}$$

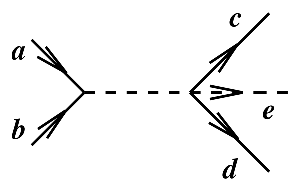
11) $= 6)\quad (c \longleftrightarrow d)$

$$= \frac{\mathrm{i}\xi^3}{\sqrt{2}\,m^4}\frac{1}{(c+c^{-1})(e+e^{-1}-\sqrt{2})} \tag{B1.11}$$

12) $= 10)\quad (c \longleftrightarrow d)$

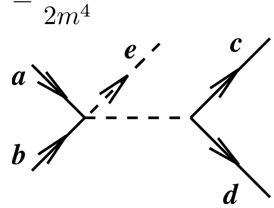
$$= \frac{\mathrm{i}\xi^3}{\sqrt{2}\,m^4}\frac{1}{(c+c^{-1})(e+e^{-1}-\sqrt{2})} \tag{B1.12}$$

13)

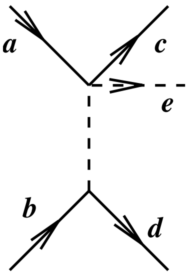


$$= \frac{\mathrm{i}\xi^3}{2m^4} \tag{B1.13}$$

14)



$$= -\frac{\mathrm{i}\xi^3}{2\sqrt{2}m^4}\frac{1}{(e+e^{-1}-\sqrt{2})} \tag{B1.14}$$

15)


$$= -\frac{\mathrm{i}\xi^3}{m^4}\frac{1}{(d+d^{-1})} \tag{B1.15}$$

16) $= 15)\ \ (a \longleftrightarrow b)$

$$= -\frac{\mathrm{i}\xi^3}{m^4}\frac{1}{(d+d^{-1})} \tag{B1.16}$$

17) $= 15)\ \ (c \longleftrightarrow d)$

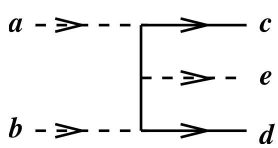
$$= -\frac{\mathrm{i}\xi^3}{m^4}\frac{1}{(c+c^{-1})} \tag{B1.17}$$

18) $= 16)\ \ (c \longleftrightarrow d)$

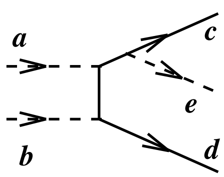
$$= -\frac{\mathrm{i}\xi^3}{m^4}\frac{1}{(c+c^{-1})} \tag{B1.18}$$

Appendix B2

We have

1)


$$= \frac{\mathrm{i}\xi^3}{m^4}\frac{1}{(2-\sqrt{2}(c+c^{-1}))}\frac{1}{(2-\sqrt{2}(d+d^{-1}))} \tag{B2.1}$$

2)


$$= -\frac{\mathrm{i}\xi^3}{4m^4}\frac{1}{(\sqrt{2}-d-d^{-1})}\frac{1}{(2\sqrt{2}-d-d^{-1})} \tag{B2.2}$$

3) $= 1)\ \ (c \longleftrightarrow d)$ (B2.3)

$$= \frac{\mathrm{i}\xi^3}{m^4}\frac{1}{(2-\sqrt{2}(c+c^{-1}))}\frac{1}{(2-\sqrt{2}(d+d^{-1}))}$$

4) $= 2)\ \ (a \longleftrightarrow b)$

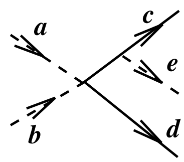
$$= -\frac{\mathrm{i}\xi^3}{4m^4}\frac{1}{(\sqrt{2}-d-d^{-1})}\frac{1}{(2\sqrt{2}-d-d^{-1})} \tag{B2.4}$$

5) $= 2)\ \ (c \longleftrightarrow d)$

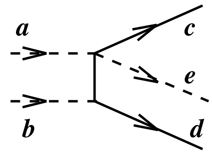
$$= -\frac{\mathrm{i}\xi^3}{4m^4}\frac{1}{(\sqrt{2}-c-c^{-1})}\frac{1}{(2\sqrt{2}-c-c^{-1})} \tag{B2.5}$$

6) $= 5)\ \ (a \longleftrightarrow b)$

$$= -\frac{\mathrm{i}\xi^3}{4m^4}\frac{1}{(\sqrt{2}-c-c^{-1})}\frac{1}{(2\sqrt{2}-c-c^{-1})} \tag{B2.6}$$

7)


$$= \frac{\mathrm{i}\xi^3}{2\sqrt{2}\,m^4}\frac{1}{(2\sqrt{2}-d-d^{-1})} \tag{B2.7}$$

8)


$$= \frac{\mathrm{i}\xi^3}{\sqrt{2}\,m^4}\frac{1}{(\sqrt{2}-d-d^{-1})} \tag{B2.8}$$

9) $= 7)\ \ (c \longleftrightarrow d)$

$$= \frac{\mathrm{i}\xi^3}{2\sqrt{2}\,m^4}\frac{1}{(2\sqrt{2}-c-c^{-1})} \tag{B2.9}$$

10) $= 8)\ \ (c \longleftrightarrow d)$

$$= \frac{\mathrm{i}\xi^3}{\sqrt{2}\,m^4}\frac{1}{(\sqrt{2}-c-c^{-1})} \tag{B2.10}$$

11) $= 8)\ \ (a \longleftrightarrow b)$

$$= \frac{\mathrm{i}\xi^3}{\sqrt{2}\,m^4}\frac{1}{(\sqrt{2}-d-d^{-1})} \tag{B2.11}$$

12) $= 11)\ \ (c \longleftrightarrow d)$

$$= \frac{\mathrm{i}\xi^3}{\sqrt{2}\,m^4}\frac{1}{(\sqrt{2}-c-c^{-1})} \tag{B2.12}$$

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